JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 5, September–October 1996

New Optimal Approach to Global Positioning System/Differential Global Positioning System Integrity Monitoring

Igor V. Nikiforov*
Université de Technologie de Troyes, Troyes Cédex 10010, France

The problems of global positioning system (GPS) and differential GPS (DGPS) integrity monitoring are addressed. Receiver autonomous and station-based integrity monitoring algorithms are discussed. Integrity monitoring consists of two functions: detection of the fact that a satellite channel produced wrong data and isolation or identification of which satellite channel is malfunctioning. So-called snapshot and fixed size sample approaches to GPS channels integrity monitoring are widely known today. Unfortunately, the snapshot and fixed size sample approaches do not lead to an optimal solution. The goal is to develop a new optimal statistical approach to GPS/DGPS channels integrity monitoring, which has significant advantages over widely known but nonoptimal snapshot and fixed size sample algorithms.

Nomenclature

$E(N)$	= expectation of the random value N
$E(\xi_t \mid W_1, \ldots, W_t)$	= conditional expectation of the random value ξ_t given $W_1, \ldots, W_t$
m	= sample size (FSS algorithm)
$\mathcal{N}(\theta, \Sigma)$	= normal law with mean vector θ and covariance matrix Σ
(N, ν)	= alarm (stopping) time and final decision
P	= distribution function
$P(B)$	= probability of the event B
$\bar{T}$	= mean time before a false alarm
$\bar{T}_{\mathrm{fa}}$	= minimum of the mean times before a false alarm
$\bar{T}_{\mathrm{fai}}$	= minimum of the mean times before a false alarm or a false isolation
$\lVert X \rVert$	= norm of X, $\sqrt{(\sum_{i=1}^{n})x_i^2}$
α	= probability of false alarm
β	= probability of miss detection
β_{fi}	= maximum of the probability of false isolation
$\bar{\tau}^*$	= worst-case mean detection delay

Subscripts

i	= current number of satellite, $1 \le i \le n$
l, j	= current number of hypothesis (fault), $0 \le l, j \le 2n$
n	= total number of visible satellites
p	= integer index
t, k, j	= time indices

I. Introduction

INTEGRITY monitoring requires that a navigation system detects and isolates faulty measurement sources, and removes them from the navigation solution before they sufficiently contaminate the output. Receiver autonomous integrity monitoring is a method of global positioning system (GPS) integrity monitoring that uses redundant GPS [or differential GPS (DGPS)] measurements at the user's receiver. Station-based integrity monitoring uses the instantaneous differences between the pseudoranges and ranges observed at a differential (monitoring) station with known locations. The GPS degradation should be detected as soon as possible when it leads to an unacceptable growth of the position errors. On the other hand, false alarms or false isolations result in lower accuracy of the estimate because some correct information is not used and, conversely, some incorrect information is used. The optimal solution involves a tradeoff between these two contradictory requirements. Numerous algorithms of GPS integrity monitoring are suggested (see Refs. 1–4). As mentioned in Ref. 1, "the majority can be described as snapshot approaches because they use a single set of GPS measurements collected simultaneously." Fixed size sample (FSS) approaches to GPS integrity monitoring are also well known in the literature. Unfortunately, the snapshot and FSS approaches do not lead to an optimal solution. We shall prove that the sequential approach presented in the paper gives the best results. The paper consists of the Introduction, four sections, and the Conclusion. Section II deals with the measurement models of GPS/DGPS signals. Sections III and IV concern the design of fault detection and detection/isolation algorithms and their statistical properties. We compare here the proposed optimal algorithms with the snapshot and FSS algorithms. Section V deals with the simulation of the algorithms and some experiments with GPS data.

II. Models of GPS and DGPS

GPS

The GPS is based on accurate measuring of the distance (range) from several satellites with known locations to a user (vehicle). Let us assume that there are n satellites located in three spaces at the known positions $X_i = (x_i, y_i, z_i)^T$, $i = 1, \ldots, n$, and a user at $X_u = (x_u, y_u, z_u)^T$ (Fig. 1). The distance from the ith satellite to the user is defined as $\tilde{d}_i = \lVert X_i - X_u \rVert$. The pseudorange $\tilde{r}_i$ from the ith satellite to the user can be written as $\tilde{r}_i = \tilde{d}_i + \tilde{b} + \tilde{\xi}_i$, $i = 1, \ldots, n$, where $\tilde{b}$ is a user clock bias and $\tilde{\xi}_i$ is an additive pseudorange error at the user position. Let us introduce the following vectors: $\tilde{R} = (\tilde{r}_1, \ldots, \tilde{r}_n)^T$ and $X = (X_u^T, \tilde{b})^T$. By linearizing the pseudorange equation with respect to the state vector X around the working point X_0, we get the measurement equation $\tilde{R} - \tilde{R}_0 \simeq H_u(X - X_0) + \tilde{\xi}$ where $\tilde{R}_0 = \tilde{R}(X_0)$, $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)^T$, $H_u = (\partial \tilde{R}/\partial X)|_{X=X_0}$ is the Jacobian matrix of size $n \times 4$. If we assume that $n \ge 4$, $E(\tilde{\xi}) = 0$ and $\mathrm{cov}(\tilde{\xi}) = \sigma^2 I_n$, where I_n is the identity matrix of order n, then the least squares algorithm provides us with an optimal navigation solution: $\hat{X} - X_0 = (H_u^T H_u)^{-1} H_u^T(\tilde{R} - \tilde{R}_0)$. Because only the selective availability degraded C/A-code signals are available for the civil users the precision of the preceding classical GPS solution is approximately 100 m (95%). For some navigation-critical applications such a precision is not sufficient, and it is necessary to apply differential techniques to remove or attenuate the effects of selective availability (SA) noise in a local area of interest.

Presented as Paper 95-3192 at the AIAA Guidance, Navigation, and Control Conference, Baltimore, MD, Aug. 7–10, 1995; received Oct. 21, 1995; revision received May 18, 1996; accepted for publication May 20, 1996.

*Professor, Département Genie des Systèmes d'Information et de Décision, Laboratoire de Modélisation et Sûreté des Systèmes, LM2S, 11, rue Marie Curie.

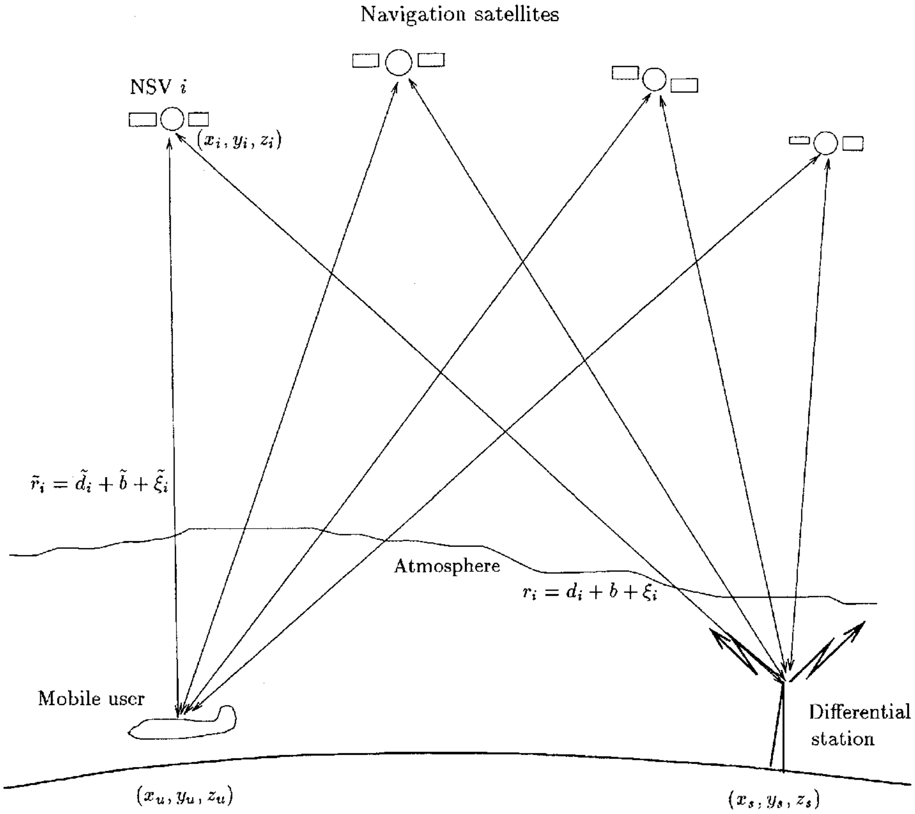

Fig. 1 DGPS.

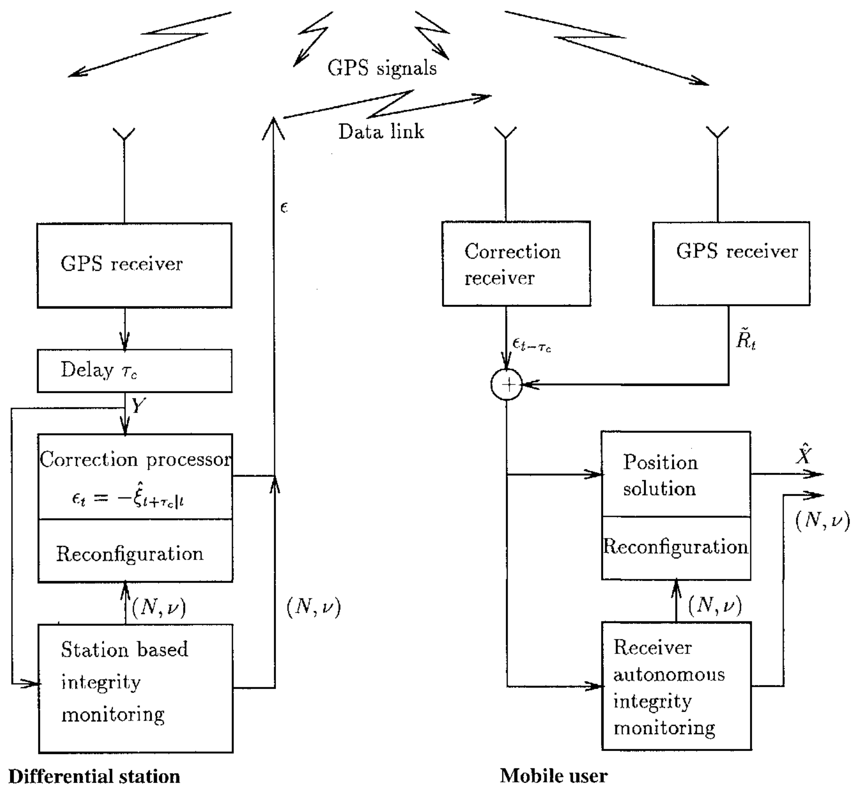

Fig. 2 Differential correction and DGPS integrity monitoring.

Differential GPS

The differential mode of GPS consists in the following: a differential station at a fixed location, whose spatial coordinates $X_s = (x_s, y_s, z_s)^T$ are precisely known, monitors the instantaneous differences between the pseudoranges $(r_1, r_2, \ldots, r_n)$ and the theoretical ones $(d_1, d_2, \ldots, d_n)$, computes and broadcasts adequate pseudorange corrections $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ to moving vehicles (see Fig. 1). In the vehicle receiver these corrections are used together with the measured signals to estimate the unknown spatial coordinates (x_u, y_u, z_u) of the vehicle (Fig. 2). This leads to elimination (or attenuation) of some important navigation satellite errors (clock and ephemeris). Nevertheless, because of the atmospheric inhomogeneity (geometric decorrelation) some residual errors remain. The distance from the ith satellite to the differential station is defined as $d_i = \|X_i - X_s\|$, $i = 1, \ldots, n$. Therefore, the vector $\boldsymbol{R}$ of pseudoranges from n satellites to the station can be written as $\boldsymbol{R} = \boldsymbol{D} + \boldsymbol{\Gamma}_n b + \boldsymbol{\xi}$, where $\boldsymbol{D} = (d_1, \ldots, d_n)^T$, $\boldsymbol{\Gamma}_n = (1, \ldots, 1)^T$, b is a station clock bias, and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$ is an additive pseudorange error at the station position. It is assumed that $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \boldsymbol{\gamma}$, where $\tilde{\boldsymbol{\xi}}$ is the pseudorange error at the station position $\boldsymbol{\xi}$ and γ is a local component of the pseudorange error at the user position; $E(\gamma_i) = 0$, $E(\gamma_i^2) = \sigma_\gamma^2$, and $E(\xi_i \gamma_i) = 0$. Let us consider the sequence $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_t$ observed at the differential station, where $\boldsymbol{W}_t = \boldsymbol{R}_t - \boldsymbol{D}_t$ is the instantaneous difference between the pseudorange $\boldsymbol{R}_t$ and the theoretical range $\boldsymbol{D}_t$, where $t = 0, 1, 2, \ldots$ is the discrete time. The problem is to compute the pseudorange correction $\epsilon_t = -\hat{\boldsymbol{\xi}}_{t+\tau_c\,|\,t}$, where $\hat{\boldsymbol{\xi}}_{t+\tau_c\,|\,t}$ is a prediction of the vector $\boldsymbol{\xi}_{t+\tau_c}$ based on $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_t$. We assume that $\tau_c \geq 1$ is a known time delay due to the computation of this correction and its transmission to the users. In the rest of the paper we will assume that $\tau_c = 1$.

SA Noise

Let us consider a GPS range bias model. It follows from Ref. 5 that the selective availability noise, which is the main component of GPS range bias, can be approximated by an autoregressive integrated moving average model with the third-order autoregressive (AR) part. The zeros of this AR-polynomial are very close to ones. Hence, the following simple model is acceptable for computing the pseudorange corrections: $\nabla^3 \boldsymbol{W}_t = \boldsymbol{\Gamma}_n \varphi_t + \boldsymbol{\psi}_t$, where $\nabla \boldsymbol{W}_t = \boldsymbol{W}_t - \boldsymbol{W}_{t-1}$, φ_t is an independent zero-mean Gaussian random sequences with variance σ_φ^2 and ψ_t is an n-dimensional independent zero-mean Gaussian random sequences with covariance $\sigma_\psi^2 I_n$.

Position Solution

Let us take into account the pseudorange correction broadcasts by the differential station (see Fig. 2): $\tilde{\boldsymbol{R}}_t + \epsilon_{t-1} = H_u \boldsymbol{X}_t + \boldsymbol{\omega}_t$, where $\tilde{\boldsymbol{R}}_t = \tilde{\boldsymbol{R}} - \tilde{\boldsymbol{R}}_0$, $\boldsymbol{X}_t = \boldsymbol{X} - \boldsymbol{X}_0$ at time instant t and $\boldsymbol{\omega}_t = \tilde{\boldsymbol{\xi}}_t + \epsilon_{t-1} = \boldsymbol{\xi}_t - \hat{\boldsymbol{\xi}}_{t|t-1} + \boldsymbol{\gamma}_t$ is an error. The optimal estimate of the user's fix $\boldsymbol{X}_u$ is given by

$$\hat{\boldsymbol{X}}_t = \left(H_u^T H_u\right)^{-1} H_u^T (\tilde{\boldsymbol{R}}_t + \epsilon_{t-1}) \tag{1}$$

Differential Corrections

It results from the theory of linear stochastic systems that the optimal (mean square) pseudorange correction is given by the following equation:

$$\epsilon_t = -\hat{\boldsymbol{\xi}}_{t+1\,|\,t} = -3\boldsymbol{W}_t + 3\boldsymbol{W}_{t-1} - \boldsymbol{W}_{t-2} \tag{2}$$

The characteristic feature of the predictor is a total compensation of additional biases, velocities, and accelerations in GPS pseudoranges. Nevertheless, the jerks and highest derivatives arising in the pseudoranges should be detected because they cannot be compensated by these differential corrections.

Models of Degradation

Under normal operating conditions, the navigation solution of the DGPS contains the useful information $\hat{\boldsymbol{X}}_t$ and the normal operation errors $\hat{\boldsymbol{X}}_t - \boldsymbol{X}_t$: $E(\hat{\boldsymbol{X}}_t - \boldsymbol{X}_t) = 0$ and $\operatorname{cov}(\hat{\boldsymbol{X}}_t - \boldsymbol{X}_t) = C_b + (H_u^T H_u)^{-1}(\sigma_\psi^2 + \sigma_\gamma^2)$, where $C_b = \operatorname{diag}(0, 0, 0, \sigma_b^2)$ and σ_b^2 is a variance of an additional time error, which is not of interest to us. We discuss only two models of DGPS degradation: global and local degradations. We assume that a global degradation leads to the same additional jerk in the pseudorange $\tilde{r}_i$ from the ith satellite to the vehicle and in the pseudorange r_i from the ith satellite to the differential station (Fig. 1). A local degradation (for example, some atmospheric anomalies in the user's area) leads to an additional bias in the pseudorange $\tilde{r}_i$ from the ith satellite to the vehicle only.

Global Degradation

Let us assume that an ith channel (global) degradation can be simulated by the following manner (see Fig. 1):

$$\nabla^3 \boldsymbol{\xi}_t = \boldsymbol{\psi}_t + \boldsymbol{\Upsilon}_l(t, t_0), \qquad \tilde{\boldsymbol{\xi}}_t = \boldsymbol{\xi}_t + \boldsymbol{\gamma}_t$$
$$l = 2i \qquad \text{or} \qquad l = 2i - 1 \tag{3}$$

where $\boldsymbol{\Upsilon}_l(t, t_0) = 0$ if $t < t_0$ and $\boldsymbol{\Upsilon}_l(t, t_0) = \boldsymbol{\Upsilon}_l$ if $t \geq t_0$. Let $\boldsymbol{\Upsilon}_l$ be $(n \times 1)$ vectors whose elements represent jerks in pseudoranges. We will assume that each vector $\boldsymbol{\Upsilon}_l = [0, \ldots, 0, (-1)^l \upsilon, 0, \ldots, 0]^T$ has only one nonzero element number i. Hence, $\boldsymbol{\Upsilon}_{l=2i-1}$ and $\boldsymbol{\Upsilon}_{l=2i}$ represent positive and negative jerks in the ith pseudorange and $\upsilon > 0$ is the absolute value of the jerk.

Local Degradation

Let us assume that an ith channel (local) degradation can be simulated by the following manner:

$$\tilde{\boldsymbol{R}}_t + \epsilon_{t-1} = H_u \boldsymbol{X}_t + \boldsymbol{\omega}_t + \tilde{\boldsymbol{\Upsilon}}_l(t, t_0) \tag{4}$$

where $\tilde{\boldsymbol{\Upsilon}}_l(t, t_0)$ is defined in the same manner as $\boldsymbol{\Upsilon}_l(t, t_0)$ but now $\tilde{\boldsymbol{\Upsilon}}_{l=2i-1}$ and $\tilde{\boldsymbol{\Upsilon}}_{l=2i}$ represent additional biases in the ith pseudorange.

Additional Bias in the User's Fix

Let us assume that a global channel degradation $\boldsymbol{\Upsilon}_l$ occurs at the unknown time t_0. Because the time delay τ_c is positive an additional bias arises in the user's fix $E(\hat{\boldsymbol{X}}_t - \boldsymbol{X}_t \mid t \geq t_0) = E(\boldsymbol{X}^0) + (H_u^T H_u)^{-1} H_u^T \boldsymbol{\Upsilon}_l$, where $\boldsymbol{X}^0 = [0, 0, 0, x^0(t, t_0)]^T$ and $x^0(t, t_0)$ is an additional clock bias, which is not of interest of us. In the case of a local degradation we simply put $\tilde{\boldsymbol{\Upsilon}}_l$ instead of $\boldsymbol{\Upsilon}_l$ in the preceding equation. Hence, the DGPS is not free from additional bias arising in the user's fix.

III. Detection of Faults in GPS/DGPS

Criterion of Fault Detection Problem

Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots,$ be a sequence of random vectors (or scalars) observed sequentially. Suppose that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}$ have distribution function P_0, whereas $\boldsymbol{Y}_{t_0}, \boldsymbol{Y}_{t_0+1}, \ldots,$ have distribution function P_1. The distributions P_0 (in control) and P_1 (out of control) are known and the change time t_0 is an unknown (but nonrandom) value. The fault detection algorithm has to compute the stopping (alarm) time N based on the observations $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots,$ the sooner the better. Two situations can occur. If the change is detected after the time t_0 ($N \geq t_0$ is true) then this detection is correct and the detection delay is $\tau = N - t_0 + 1$. On the other hand, if the change in P is detected before the time t_0 ($N < t_0$ is true) it corresponds to a false alarm. It is obvious that the criterion that must be used favors fast detection with few false alarms. In other words, $\tau \mid N \geq t_0$ should be stochastically small and $N \mid N < t_0$ should be stochastically large. Let P_{t_0} denote the common distribution function of the observations $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots,$ when $\boldsymbol{Y}_{t_0}$ is the first observation with distribution P_1 and E_{t_0} denotes the expectation under P_{t_0}. We require that the worst-case mean detection delay

$$\bar{\tau}^* = \sup_{t_0 \geq 1} \operatorname{esssup} E_{t_0}\left(N - t_0 + 1 \mid N \geq t_0, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}\right) \tag{5}$$

should be as small as possible for a given mean time $\bar{T}$ before a false alarm

$$\bar{T} = E_0(N) \tag{6}$$

where the essential supremum of the conditional expectation $\bar{\tau} = E_{t_0}(N - t_0 + 1 \mid N \geq t_0, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1})$ is defined by $\operatorname{esssup} E_{t_0} (N - t_0 + 1 \mid N \geq t_0, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}) \equiv \inf\{0 \leq c \leq \infty \colon P(\bar{\tau} > c) = 0\}$ and E_0 denotes the expectation under P_0. Therefore, the statistical properties of detection algorithms are defined with the aid of the worst-case mean detection delay $\bar{\tau}^*$ and the mean time $\bar{T}$ before a false alarm.

Remark 1. Let us discuss the optimality criterion introduced by Lorden.[6] It is obvious that, without knowing a priori the distribution of the change time t_0, the mean detection delay is a function of the change time t_0 and of the past values $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}$ of the random sequence. In many practical cases it is very useful to have an algorithm that is independent of the distribution of the change time t_0 and the sample path of the past observations $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}$. For this reason we use the minimax criteria (5) and (6), which consist of minimizing the worst-case mean detection delay for a given mean time before a false alarm.

Simple Example

Model

Let $y_1, y_2, \ldots,$ be an independent scalar Gaussian random sequence. Suppose that $y_1, \ldots, y_{t_0-1}$ have distribution $P_0 = \mathcal{N}(\theta_0, \sigma^2)$ whereas $y_{t_0}, y_{t_0+1}, \ldots,$ have distribution $P_1 = \mathcal{N}(\theta_1, \sigma^2)$ where θ_0, θ_1, and σ^2 are known a priori. Let us now briefly describe two candidates for the comparison.

Optimal Sequential Strategy

The idea of the cumulative sum (CUSUM) algorithm proposed by Page[7] is to set the alarm at the first instant N_1 for which the decision function g_t reaches a threshold h that is selected a priori:

$$N_1 = \inf\{t \geq 1 : g_t \geq h\}, \qquad g_t = \left[g_{t-1} + \ln \frac{p_{\theta_1}(y_t)}{p_{\theta_0}(y_t)}\right]^+$$

$$p_\theta(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\theta)^2}{2\sigma^2}\right\}$$

where $g_0 = 0$, $(x)^+ = \max(x, 0)$, $\ln[p_{\theta_1}(y_t)/p_{\theta_0}(y_t)]$ is the log-likelihood ratio for the observation y_t, and $p_{\theta_0}(y_t)$ and $p_{\theta_1}(y_t)$ are the probability densities corresponding to P_0 and P_1.

Optimal FSS Strategy

The optimal FSS strategy is based on the rule that samples with fixed size m are taken, and at the end of each sample a decision function of the Neyman–Person test is computed to test between the hypotheses $\mathcal{H}_0$: $\theta = \theta_0$ and $\mathcal{H}_1$: $\theta = \theta_1$. Sampling is stopped after the first sample of observations for which the decision is taken in favor of $\mathcal{H}_1$:

$$\bar{N}_1 = \inf_{j \geq 1}\{mj : \bar{g}_j \geq \bar{h}\}, \qquad \bar{g}_j = \sum_{t=(j-1)m+1}^{jm} \ln \frac{p_{\theta_1}(y_t)}{p_{\theta_0}(y_t)} \tag{7}$$

where $\bar{g}_j$ is the log-likelihood ratio corresponding to the jth sample of size m and $\bar{h}$ is a conveniently chosen threshold.

Comparison

The optimality theorems for these algorithms can be found in Refs. 6 and 8. Asymptotically, when the mean time $\bar{T}$ before a false alarm goes to infinity the properties of these procedures are given by the following relations.

CUSUM:

$$\bar{\tau}^* \sim \frac{\ln \bar{T}}{\rho(\theta_1, \theta_0)} \tag{8}$$

FSS:

$$\bar{\tau}^* \sim 2\frac{\ln \bar{T}}{\rho(\theta_1, \theta_0)} \tag{9}$$

where

$$\rho(\theta_1, \theta_0) = E_{\theta_1}\left[\ln \frac{p_{\theta_1}(y_t)}{p_{\theta_0}(y_t)}\right] = \frac{(\theta_1 - \theta_0)^2}{2\sigma^2}$$

is the Kullback–Leibler information.

Remark 2. In the classical theory of testing between two hypotheses the quality of a statistical test is usually defined with the aid of the probability of false alarm α and the probability of miss detection β. Some previous works (see Refs. 1 and 2) focused on the problem of navigation systems integrity monitoring used these performance indices to estimate the quality of FSS decision procedures. We emphasize that the probabilities α and β by themselves are not reasonable performance indices in the case of the on-line change detection. It is necessary to take into account the period (sample size m) of the separate stages of detection. Ignoring this fact can lead to loss of efficiency of the fault detection algorithms. From Ref. 8 it follows that the optimal values of the probability of false alarm α and the probability of miss detection β are given by the following asymptotic formulas:

$$\alpha^* \sim \frac{1}{\sqrt{2\pi}\bar{T}\sqrt{2\ln \bar{T}}} \quad \text{and} \quad \beta^* \sim \frac{1}{\sqrt{2\pi \ln \bar{T} \cdot \ln \ln \bar{T}}}$$

as $\bar{T} \to \infty$. On the other hand, the optimal choice of the tuning parameters h, m is: $h^* \sim \ln \bar{T}$ and $m^* \sim [\ln \bar{T}/\rho(\theta_1, \theta_0)]$ as $\bar{T} \to \infty$.

Design of the Algorithms

Models

It follows from Sec. II that the detection of global and local degradations in a channel (or channels) of GPS/DGPS is reduced to the detection of Υ in the measurement models,

$$Y_t = \nabla^3 W_t = \Gamma_n \varphi_t + \psi_t + \Upsilon(t, t_0) \tag{10}$$

$$Y_t = \tilde{R}_t + \epsilon_{t-1} = H_u X_t + \omega_t + \tilde{\Upsilon}(t, t_0) \tag{11}$$

where $\Upsilon(t, t_0) = 0$ if $t < t_0$ and $\Upsilon(t, t_0) = \Upsilon$ if $t \geq t_0$; $\tilde{\Upsilon}(t, t_0)$ is defined in the same manner, ω_t is an n-dimensional independent zero-mean Gaussian random sequence with covariance cov $= (\sigma_\varphi^2 + \sigma_y^2) I_n$. We omit now the only one fault assumption. Several satellite channels can fail at a time. Let us start with model (10).

Station-Based Integrity Monitoring

It is clear that $Y_1, Y_2, \ldots,$ is an n-dimensional independent Gaussian random sequence, where $Y_1, \ldots, Y_{t_0-1}$ have distribution $P_0 = \mathcal{N}(0, \Sigma)$ whereas $Y_{t_0}, Y_{t_0+1}, \ldots,$ have distribution $P_1 = \mathcal{N}(\Upsilon, \Sigma)$, where $\Upsilon \in \Theta$. It is easy to see that $\Sigma = \Gamma_n \Gamma_n^T \sigma_\varphi^2 + \sigma_\psi^2 I_n$. The vector $\Upsilon \in \Theta$ is unknown but the set Θ is defined by the ellipsoid $\Upsilon^T \Sigma^{-1} \Upsilon = b^2$, where b is a signal-to-noise ratio (SNR). In other words, we assume that the direction of change is unknown but the magnitude is known.

Optimal Sequential Strategy

Let us define the stopping time N_n of the χ^2-CUSUM algorithm[9,10]

$$N_n = \inf\left\{t \geq 1 : \max_{1 \leq k \leq t} S_k^t \geq h\right\} \tag{12}$$

$$S_k^t = -(t-k+1)\frac{b^2}{2} + \ln G\left[\frac{n}{2}, \frac{b^2(t-k+1)^2\left(\chi_k^t\right)^2}{4}\right] \tag{13}$$

$$\left(\chi_k^t\right)^2 = \left(\bar{Y}_k^t\right)^T \Sigma^{-1} \bar{Y}_k^t, \qquad \bar{Y}_k^t = (t-k+1)^{-1} \sum_{j=k}^{t} Y_j \tag{14}$$

$$\Sigma^{-1} = \left\{I_n - \left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)\right]^{-1} \Gamma_n \Gamma_n^T\right\} \sigma_\psi^{-2} \tag{15}$$

where $S_k^t = S_k^t(Y_k, \ldots, Y_t)$ is the log-likelihood ratio for testing that the noncentrality parameter of a χ^2 distribution with n degrees of freedom is either equal to 0 or to $(t-k+1)b^2$; the quadratic form $(\chi_k^t)^2$ and the mean $\bar{Y}_k^t$ are computed for the observations $Y_k, \ldots, Y_t$, where subscript k and superscript t denote the first and the last observations of the time window, h is a threshold, and

$$G(\kappa, x) = 1 + \frac{x}{\kappa} + \frac{x^2}{\kappa(\kappa+1)2!} + \cdots + \frac{x^p}{\kappa(\kappa+1)\cdots(\kappa+p-1)p!} + \cdots$$

is the generalized hypergeometric function where $\kappa = n/2$. The SNR b and the threshold h are the tuning parameters of the χ^2-CUSUM algorithm.

Optimal FSS Strategy

The alarm time $\bar{N}_n$ of the repeated χ^2-FSS test applied to samples of the fixed size m is

$$\bar{N}_n = \inf_{j \geq 1}\left\{mj : \left(\chi_{(j-1)m+1}^{jm}\right)^2 \geq h^2\right\} \tag{16}$$

where $(\chi_{(j-1)m+1}^{jm})^2$ is defined in Eq. (14). The window size m and the threshold h are the tuning parameters of the χ^2-FSS algorithm.

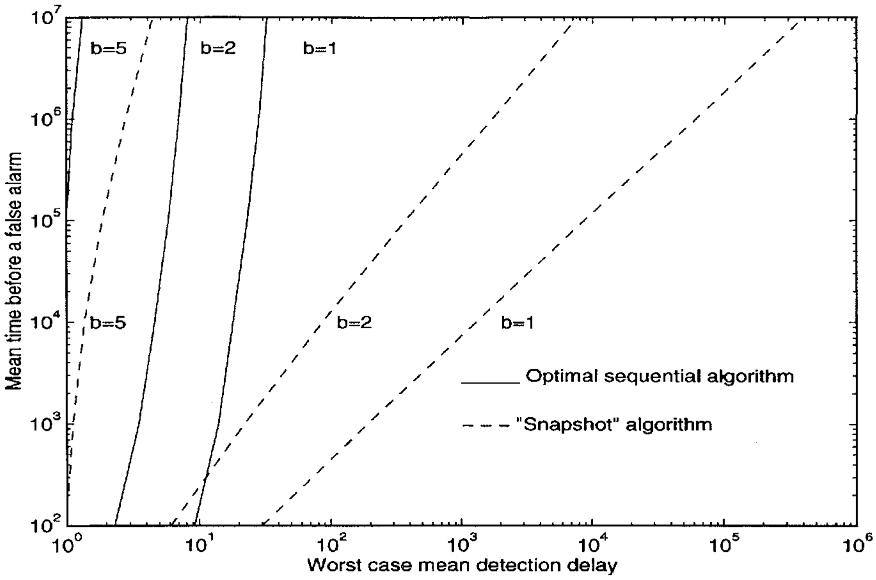

Fig. 3 Comparison between the χ^2-CUSUM and snapshot algorithms: the mean time before a false alarm $\bar{T}$ of the χ^2-CUSUM algorithm and the mean time before a false alarm $\bar{T}$ of the snapshot algorithm are plotted as functions of the worst-case mean detection delay $\bar{\tau}^*$.

Snapshot Algorithm

Let us consider the χ^2-FSS algorithm when the sample size is $m = 1$. This algorithm is called snapshot. The alarm rule of the snapshot is given by

$$\tilde{N}_n = \inf\left\{t \geq 1 : Y_t^T \Sigma^{-1} Y_t \geq h^2\right\} \tag{17}$$

Algorithms (16) and (17) are widely used in the area of navigation system integrity monitoring (see Refs. 1–3).

Statistical Properties of the Algorithms

The optimality theorem and the statistical properties of the χ^2-CUSUM algorithm can be found in Refs. 9 and 10:

$$\bar{\tau}^* \sim \frac{\ln \bar{T}}{\rho(\theta_1, \theta_0)} \quad \text{as} \quad \bar{T} \to \infty \tag{18}$$

where $\rho(\theta_1, \theta_0) = b^2/2$. The statistical properties of the χ^2-FSS algorithm are given by

$$\bar{\tau}^*(m, h) = \max\left\{\frac{m}{1-\beta}, \max_{1<l\leq m}\left[m - l + 1 + \frac{m}{1-\beta} P\left(\chi^2_{n,0} < \frac{m^2h^2}{m-l+1}\right)\right]\right\} \tag{19}$$

where $\beta = P(\chi^2_{n,mb^2} < mh^2)$ and $\chi^2_{n,\lambda}$ is a random value distributed according to the noncentral χ^2 law with n degrees of freedom and the noncentrality parameter λ,

$$\bar{T}(m, h) = m/\alpha, \qquad \alpha = 1 - P\left(\chi^2_{n,0} < mh^2\right) \tag{20}$$

It follows from Eqs. (19) and (20) that the worst-case mean detection delay $\bar{\tau}^*$ is a function of the tuning parameters m and h of algorithm (16) for a given value of $\bar{T}$. Therefore, the optimal choice of the parameters m and h is reduced to minimizing the worst-case mean detection delay $\bar{\tau}^*(m, h)$ under the constraint $\bar{T}(m, h) = \text{const}$. Let us name this algorithm optimal χ^2-FSS. Obviously, in the case of the snapshot algorithm the sample size m is equal to 1 and, hence, $\bar{\tau}^* = \bar{\tau}^*(\bar{T})$.

Receiver Autonomous Integrity Monitoring

Let us consider model (11) now. Let us assume that $n \geq 5$ satellites are visible and the matrix H_u has rank 4. The characteristic feature of this model is that X_t is an unknown (but nonrandom) state vector. Such statistical tasks are usually called hypotheses testing problems with nuisance parameters. (See the tutorial introduction to these problems in Ref. 10, pp. 141–145 and 270–273.) The optimal approach consists of finding a pair of the least favorable values X_t^0 and X_t^1 of the unknown state X_t under two alternative hypotheses that minimize the Kullback–Leibler information between these hypotheses.[10] To solve the detection problem it is necessary to transform the observations $Y_1, Y_2, \ldots, Y_t$ into the residual errors $e_1, e_2, \ldots, e_t$ of the least square algorithm and then to apply one of the aforementioned algorithms. It follows from Eq. (1) that the residual vector e_t is given by $e_t = \Pi Y_t$, where $\Pi = I_n - H_u(H_u^T H_u)^{-1} H_u^T$. Equations (18–20) are also valid in the case of receiver autonomous integrity monitoring. Obviously, the noncentrality parameter b^2 is equal to $(\sigma_\varphi^2 + \sigma_\gamma^2)^{-1} \tilde{\Upsilon}^T \Pi \tilde{\Upsilon}$. The χ^2-CUSUM algorithm for model (11) is obtained by replacing n, $\bar{Y}_i^k$, and Σ^{-1} in Eqs. (12–14), by $(n-4)$,

$$\bar{e}_k^t = (t - k + 1)^{-1} \sum_{j=k}^{t} e_j$$

and $(\sigma_\varphi^2 + \sigma_\gamma^2)^{-1}\Pi$. The snapshot algorithm for model (11) is obtained by replacing Y_t and Σ^{-1} in Eq. (17), by e_t and $(\sigma_\varphi^2 + \sigma_\gamma^2)^{-1}\Pi$.

χ^2 CUSUM vs Snapshot

The comparison was performed for $b = 1, 2$, and 5. We assume that in the case of station-based detection there are three visible satellites, and in the case of receiver autonomous detection seven satellites are visible. The results of this comparison are given in Fig. 3. This figure shows that the worst-case mean delay $\bar{\tau}^*$ of the χ^2-CUSUM algorithm is significantly lower then the worst-case mean delay $\bar{\tau}^*$ of the snapshot algorithm, especially for small values of the SNR b. For example, let us assume that the sampling period is equal to 0.6 s; if $b = 1$ and $\bar{T} = 10^7 \sim 0.2$ year, then the mean delay for detection of the χ^2-CUSUM algorithm is equal to $\simeq$20 s but the mean delay for detection of the snapshot algorithm is equal to $\simeq$64.8 h! The essential advantage of the χ^2-CUSUM algorithm over the snapshot approach is explained by averaging the multiple observations over a sliding window with random size. Nevertheless, in the case of a large SNR ($b \geq 5$) and a relatively small $\bar{T}$ the delays for the detection are almost the same for both algorithms!

χ^2 CUSUM vs Optimal χ^2 FSS

Note here that 1) exactly as in a scalar case the χ^2-CUSUM algorithm is twice as good as the nonsequential competitor; and 2) the correct choice of m plays a key role in the performance of the χ^2-FSS algorithm (see also Remark 2).

IV. Detection/Isolation of Faults in GPS/DGPS

Criteria of Fault Detection/Isolation Problem

Suppose now that the observations $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}$ have distribution function P_0, whereas $\boldsymbol{Y}_{t_0}, \boldsymbol{Y}_{t_0+1}, \ldots,$ have distribution function P_l. The distributions P_0 (in control) and $P_l, l = 1, \ldots, 2n$ (out of control, fault number l), are known, but the change time t_0 and number l are unknown. The fault detection/isolation algorithm has to compute a pair of two integers (N, ν) where N is the alarm time at which a ν-type fault is identified. Let $P_{t_0}^l$ denote the common distribution function of the observations $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots,$ when $\boldsymbol{Y}_{t_0}$ is the first observation with distribution P_l, and $E_{t_0}^l$ denotes the expectation under $P_{t_0}^l$. Here, superscript l denotes that fault number l arises at the instant t_0. The following three situations can occur: 1) correct detection, i.e., $N > t_0$ and $\nu = l$; 2) false alarm, i.e., $N < t_0$; and 3) false isolation, i.e., $N \geq t_0$ and $\nu \neq l$. The detection delay $\tau = N - t_0 + 1$ is defined in Sec. III. Let the observations $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_t$ have distribution P_0. Consider the sequence of false alarms $N_1, \ldots, N_r, \ldots,$ where N_r is the alarm time of the detection/isolation algorithm, which is applied to the observations $\boldsymbol{Y}_{N_{r-1}+1}, \boldsymbol{Y}_{N_{r-1}+2}, \ldots$. The first false alarm time of a j type in this sequence is $\inf_{r\geq 1}(N_r : \nu_r = j)$. Let the observations $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_t$ have distribution P_l (fault number l). The first false isolation time of a j type in the sequence of false isolations is $\inf_{r\geq 1}(N_r : \nu_r = j)$. It is intuitively obvious that the detection delay $\tau \mid N \geq t_0$ should be stochastically small for each $l = 1, \ldots, 2n$, and $(N_r : \nu_r = j \neq l)$ should be stochastically large for each combinations of numbers $j \neq l$.

Criterion A

We require that the worst-case mean detection/isolation delay

$$\bar{\tau}^* = \max_{1\leq l\leq 2n} \sup_{t_0\geq 1} \operatorname{esssup} E_{t_0}^l \left(N - t_0 + 1 \mid N \geq t_0, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{t_0-1}\right) \tag{21}$$

should be as small as possible for a given minimum $\bar{T}$ of the mean times before a false alarm or a false isolation

$$\min_{0\leq l\leq 2n} \min_{1\leq j\neq l\leq 2n} E_l\left(\inf_{r\geq 1}\{N_r : \nu_r = j\}\right) = \bar{T}_{\text{fai}} \tag{22}$$

where $E_l(\ldots)$ is the expectation under the distribution P_l of the observation $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_t$.

Criterion B

We require that the worst-case mean detection/isolation delay $\bar{\tau}^*$ Eq. (21) should be as small as possible subject to the following inequalities constraints:

$$\min_{1\leq l\leq 2n} E_0\left(\inf_{r\geq 1}\{N_r : \nu_r = l\}\right) = \bar{T}_{\text{fa}} \tag{23}$$

$$\max_{1\leq l\leq 2n} \max_{1\leq j\neq l\leq 2n} P_l(\nu = j \neq l) = \beta_{\text{fi}} \tag{24}$$

where $\bar{T}_{\text{fa}}$ is a given minimum of the mean times before a false alarm and β_{fi} is a given maximum value of the probability of false isolation.

Design of the Algorithms

Station-Based Integrity Monitoring

Let us start again with model (3). The global fault of an ith channel is

$$\boldsymbol{Y}_t = \nabla^3 \boldsymbol{W}_t = \boldsymbol{\Gamma}_n \varphi_t + \psi_t + \boldsymbol{\Upsilon}_l(t, t_0)$$
$$l = 2i - 1 \quad \text{or} \quad l = 2i \tag{25}$$

where $\boldsymbol{\Upsilon}_l(t, t_0) = 0$ if $t < t_0$ and $\boldsymbol{\Upsilon}_l(t, t_0) = \boldsymbol{\Upsilon}_l$ if $t \geq t_0$, $\boldsymbol{\Upsilon}_l = [0, \ldots, 0, (-1)^l \upsilon, 0, \ldots, 0]^T, \upsilon > 0$. The ith elements $(-1)^l \upsilon$ represent positive or negative jerks in the ith pseudorange. We assume that n satellites are visible, hence, there are $2n$ different faults but only one satellite channel can fail at a time.

Optimal Sequential Strategy

Mathematical results and details on the sequential fault detection/isolation algorithm have been described previously in Refs. 11 and 12. For this reason we start directly with the alarm time and the final decision of the station-based fault detection/isolation algorithm

$$N = \min\{N^1, \ldots, N^{2n}\}, \qquad \nu = \operatorname{argmin}\{N^1, \ldots, N^{2n}\} \tag{26}$$

This decision making rule means that there are $2n$ simultaneous alarm rules $N^1, \ldots, N^{2n}$. The satellite channel corresponding to the rule that is stopped first (this is denoted by using the argument of the $\min\{N^1, \ldots, N^{2n}\}$) is identified as faulty. The stopping time N^l corresponding to fault number l is defined in the following manner:

$$N^l = \inf\left\{t \geq 1 : \max_{1\leq k\leq t} \min_{0\leq j\neq l\leq 2n} \left[S_k^t(l, j) - h_{lj}\right] \geq 0\right\} \tag{27}$$

$$S_k^t(l, j) = S_k^t(l, 0) - S_k^t(j, 0)$$

$$S_k^t(l, 0) = \sum_{j=k}^{t} \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{Y}_j - \frac{t + k - 1}{2} \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{\Upsilon}_l$$

where $S_k^t(l, j) = S_k^t(l, j, \boldsymbol{Y}_k, \ldots, \boldsymbol{Y}_t)$ is the log-likelihood ratio for testing that $\boldsymbol{\Upsilon}$ is either equal to $\boldsymbol{\Upsilon}_j$ or to $\boldsymbol{\Upsilon}_l$, subscript k and superscript t denote the first and the last observations of the time window, h_{lj} are a priori chosen thresholds, $1 \leq l \leq 2n$, $0 \leq j \neq l \leq 2n$, $\boldsymbol{\Upsilon}_0 = 0$, and Σ^{-1} is defined in Eq. (15) In the case of criterion A we choose the same thresholds $h_{lj} = h$, but in the case of criterion B the thresholds are chosen by the following formula:

$$h_{lj} = \begin{cases} h_d & \text{if} \quad l = 1, \ldots, 2n \quad \text{and} \quad j = 0 \\ h_i & \text{if} \quad j, l = 1, \ldots, 2n \quad \text{and} \quad j \neq l \end{cases} \tag{28}$$

where h_d is the detection threshold and h_i is the isolation threshold.

Optimal FSS Strategy

The idea of a FSS change detection/isolation strategy is based on the rule that samples with fixed size m are taken, and at the end of each sample a decision function is computed to test between the hypotheses $\mathcal{H}_0 : \boldsymbol{\Upsilon} = 0, \mathcal{H}_1 : \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1, \ldots, \mathcal{H}_{2n} : \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_{2n}$. Sampling is stopped after the first sample of observations for which the decision $\tilde{\nu}$ is taken in favor of $\mathcal{H}_{\tilde{\nu}} : \tilde{\nu} > 0$. (See details in Ref. 8.) The stopping time $\tilde{N}$ and the final solution ν of the FSS change detection/isolation algorithm are given by

$$\tilde{N} = \inf_{j\geq 1}\left\{jm : \max_{1\leq l\leq 2n} S_j(l, 0) \geq h\right\} \tag{29}$$

$$\tilde{\nu} = \arg\max_{1\leq l\leq 2n} \{S_j(l, 0) : S_j(l, 0) > h\} \tag{30}$$

$$S_j(l, 0) = \sum_{t=(j-1)m+1}^{jm} \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{Y}_t - \frac{m}{2} \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{\Upsilon}_l \tag{31}$$

where $S_j(l, 0) = S_j(l, 0, \boldsymbol{Y}_{(j-1)m+1}, \ldots, \boldsymbol{Y}_{jm})$ is the log-likelihood ratio of the jth sample of size m for testing that $\boldsymbol{\Upsilon}$ is either equal to 0 or to $\boldsymbol{\Upsilon}_l$. The sample size m and the threshold h are the tuning parameters of the FSS fault detection/isolation algorithm.

Snapshot Algorithm

Let us consider the preceding FSS fault detection/isolation algorithm when the sample size is $m = 1$. This detection/isolation algorithm is called snapshot. The alarm time $\tilde{N}$ and the final decision $\tilde{\nu}$ of the snapshot algorithm are given by

$$\tilde{N} = \inf\left\{t \geq 1 : \max_{1\leq l\leq 2n} S_t(l, 0) \geq h\right\} \tag{32}$$

$$\tilde{\nu} = \arg\max_{1\leq l\leq 2n} \{S_t(l, 0) : S_t(l, 0) > h\} \tag{33}$$

$$S_t(l, 0) = \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{Y}_t - \tfrac{1}{2} \boldsymbol{\Upsilon}_l^T \Sigma^{-1} \boldsymbol{\Upsilon}_l \tag{34}$$

Remark 3 (discussion of the snapshot algorithm). Let us now show that the preceding snapshot algorithm can be also interpreted as some reasonable heuristic rules, e.g., an approach of comparing each pseudorange with the solution that is obtained without using that pseudorange. We assume that no a priori information on the model of station clock errors exists, that is, $\sigma_\varphi^2 \to \infty$ and that $\boldsymbol{Y}_t = \boldsymbol{R}_t - \boldsymbol{D}_t$ is the instantaneous difference between the pseudorange $\boldsymbol{R}_t$ and the theoretical range $\boldsymbol{D}_t$. It follows from Eq. (34) that the log-likelihood ratio $S_t(l, 0)$ is given by the following simple formula:

$$S_t(l,0) = \frac{(-1)^l \upsilon}{\sigma_\psi^2}\left[y_{t,i} - \frac{1}{n}(y_{t,1} + \cdots + y_{t,n})\right] - \frac{\upsilon^2}{2\sigma_\psi^2}\left(1 - \frac{1}{n}\right)$$

where $y_{t,i}$ is the ith element of the vector $\boldsymbol{Y}_t$ and σ_ψ^2 is the pseudorange error variance. Because n, σ_ψ^2, and υ are known, the detection/isolation rule of the snapshot algorithm consists in computing the residuals δ_i for the n subsets of $n-1$ satellites (the ith satellite is omitted)

$$\delta_i = y_{t,i} - \frac{1}{n-1}\sum_{p \neq i}^{n} y_{t,p}$$

and choosing $\max(|\delta_1|, |\delta_2|, \ldots, |\delta_n|)$. If the preceding maximum is greater than a selected threshold, a fault has been detected. The argument of this maximum shows which satellite channel is faulty. It is easy to see that the preceding snapshot algorithm is also equivalent to the other widely used decision making rule,[1,3] which consists in computing the sums of squares of the residual errors (*SSE*)

$$SSE_i = \sum_{p \neq i}^{n} (y_{t,p} - \delta_i)^2$$

for the n subsets of $n-1$ satellites (the ith satellite is omitted) and in identifying the satellite channel omitted from the subset that gives the smalles SSE_i as the failed one.

Statistical Properties of the Algorithms

The statistical properties and optimality of algorithm of the sequential detection/isolation algorithm (26) and (27) has been investigated in Refs. 11 and 12. The asymptotic relation between $\bar{\tau}^*$ and $\bar{T}_{\text{fai}}$ (criterion A) is given by

$$\bar{\tau}^* \sim \frac{\ln \bar{T}_{\text{fai}}}{\rho^*} \qquad \text{as} \qquad \bar{T}_{\text{fai}} \to \infty \tag{35}$$

where

$$\rho^* = \begin{cases} \left\{1 - \left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)\right]^{-1}\right\}\left(\upsilon^2/2\sigma_\psi^2\right) \\ \qquad\qquad \text{if} \quad 1 - 3\left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)^{-1}\right] \geq 0 \\ \left\{1 - 2\left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)\right]^{-1}\right\}\left(\upsilon^2/\sigma_\psi^2\right) \\ \qquad\qquad \text{if} \quad 1 - 3\left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)^{-1}\right] < 0 \end{cases}$$

and

$$\rho_{lj} = E_l\left[\ln \frac{p_{\Upsilon_l}(\boldsymbol{Y}_t)}{p_{\Upsilon_j}(\boldsymbol{Y}_t)}\right]$$

is the Kullback–Leibler information between the hypotheses $\mathcal{H}_l$ and $\mathcal{H}_j$. Let us turn now to criterion B. The asymptotic relation between $\bar{\tau}^*$, $\bar{T}_{\text{fa}}$, and β_{fi} is given by

$$\bar{\tau}^* \sim \max\left\{\frac{\ln \bar{T}_{\text{fa}}}{\rho_{\text{fa}}^*}, \frac{\ln\left(\bar{\tau}^* \beta_{\text{fi}}^{-1}\right)}{\rho_{\text{fi}}^*}\right\}$$

$$\text{as} \quad \bar{T}_{\text{fa}} \to \infty, \qquad \beta_{\text{fi}} \to 0, \qquad \bar{T}_{\text{fa}}\beta_{\text{fi}} = \text{const} \tag{36}$$

$$\rho_{\text{fa}}^* = \left\{1 - \left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)\right]^{-1}\right\}\left(\upsilon^2/2\sigma_\psi^2\right)$$

$$\rho_{\text{fi}}^* = \left\{1 - 2\left[n + \left(\sigma_\psi^2/\sigma_\varphi^2\right)\right]^{-1}\right\}\left(\upsilon^2/\sigma_\psi^2\right)$$

Some simplifications of the problem should be done so that the statistical properties of the FSS algorithm (29–31) can be obtained. We assume that $(\sigma_\psi/\sigma_\varphi) \gg 1$ and the sign of υ is known a priori. Under these assumptions, the statistical properties of the FSS algorithms in the sense of criteria A and B are given. (See details in Ref. 8.) Namely, the worst-case mean detection delay is

$$\bar{\tau}^* = \frac{\sigma_\psi^2(x-y)^2}{\upsilon^2\{1 - [1 - \Phi(y)][1 - \Phi(x)]^{n-2}\}} + \frac{\sigma_\psi^2(x-y)^2}{\upsilon^2} - 1$$

$$x = (h + m_\rho)/\sqrt{2m_\rho}, \qquad y = (h - m_\rho)/\sqrt{2m_\rho} \tag{37}$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-(x^2/2)}\mathrm{d}x, \qquad m_\rho = m\frac{\upsilon^2}{2\sigma_\psi^2}$$

The mean time before a false alarm is given by

$$\bar{T}_{\text{fai}} = \bar{T}_{\text{fa}} = \frac{\sigma_\psi^2(x-y)^2(n-1)}{\upsilon^2\{1 - [1 - \Phi(x)]^{n-1}\}} \tag{38}$$

Bounds for the probability of false isolation are given by

$$\frac{\Phi(x)[1 - \Phi(y)][1 - \Phi(x)]^{n-3}}{1 - [1 - \Phi(y)][1 - \Phi(x)]^{n-2}}$$

$$\leq \beta_{\text{fi}} \leq \frac{1 - [1 - \Phi(x)]^{n-1}}{(n-1)\{1 - [1 - \Phi(y)][1 - \Phi(x)]^{n-2}\}} \tag{39}$$

It follows from the preceding equations that the worst-case mean detection delay $\bar{\tau}^*$ and the probability of false isolationis β_{fi} are functions of the tuning parameters m and h of the FSS algorithm for given values of $\bar{T}_{\text{fa}}$. Therefore, the optimal choice of the parameters m and h can be reduced to minimizing the worst-case mean detection delay $\bar{\tau}^*(m, h)$ under the constraint $\bar{T}(m, h) = \text{const}$.

Receiver Autonomous Integrity Monitoring

Let us now consider model (4) with n visible satellites and $2n$ types of faults. The local fault of an ith channel is

$$\boldsymbol{Y}_t = \tilde{\boldsymbol{R}}_t + \epsilon_{t-1} = H_u \boldsymbol{X}_t + \omega_t + \tilde{\boldsymbol{\Upsilon}}_l(t, t_0)$$

$$l = 2i - 1 \qquad \text{or} \qquad l = 2i \tag{40}$$

where $\tilde{\boldsymbol{\Upsilon}}_l(t, t_0)$ is defined in Eq. (25) and $\tilde{\boldsymbol{\Upsilon}}_l = [0, \ldots, 0, (-1)^l \upsilon_i, 0, \ldots, 0]^T$, $\upsilon_i > 0$. To obtain a stable detection/isolation of faults it is necessary to choose the tuning value of the fault magnitude of the ith satellite in the following manner when $\upsilon_i = \tilde{\upsilon}/\sqrt{p_{ii}}$, $\tilde{\upsilon} > 0$. It is necessary to transform the observations $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_t$ into the residuals $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_t$ of the least square algorithm, where $\boldsymbol{e}_t = \Pi \boldsymbol{Y}_t$ and then to apply one of the aforementioned algorithms (sequential or FSS). See details and proofs in Refs. 11 and 12. Hence, the log-likelihood ratio $S_k^t(l, j)$ is given by

$$S_k^t(l, j) = S_k^t(l, 0) - S_k^t(j, 0)$$

$$S_k^t(l, 0) = \sum_{j=k}^{t}\left[\frac{(-1)^l \tilde{\upsilon} e_{j,i}}{\left(\sigma_\varphi^2 + \sigma_\gamma^2\right)\sqrt{p_{ii}}} - \frac{\tilde{\upsilon}^2}{2\left(\sigma_\varphi^2 + \sigma_\gamma^2\right)}\right] \tag{41}$$

where $i = [(l+1)/2]$ is the integral part of $(l+1)/2$, $l = 1, \ldots, 2n$, $e_{t,i}$ is the ith component of the residual vector $\boldsymbol{e}_t$, and p_{ii} is a diagonal element of the matrix Π. Equations (35) and (36) are valid for the sequential fault detection/isolation algorithm in the case of receiver autonomous integrity monitoring when $\upsilon_i = \tilde{\upsilon}/\sqrt{p_{ii}}$. The minimum values of the Kullback–Leibler information are given by the following equations:

$$\rho^* = \min\left(\rho_{\text{fa}}^*, \rho_{\text{fi}}^*\right), \qquad \rho_{\text{fa}}^* = \frac{\tilde{\upsilon}^2}{2\left(\sigma_\varphi^2 + \sigma_\gamma^2\right)}$$

$$\rho_{\text{fi}}^* = \frac{\tilde{\upsilon}^2}{\sigma_\varphi^2 + \sigma_\gamma^2}\min_{1 \leq l \leq n}\ \min_{1 \leq j \neq l \leq n}\ \min\left(1 \pm \frac{p_{jl}}{\sqrt{p_{ll}p_{jj}}}\right)$$

where p_{ll}, p_{jj}, and p_{jl} are elements of the matrix Π. Unfortunately, it is difficult to obtain analytical relations between $\bar{\tau}^*$, $\bar{T}_{\text{fi}}$, and β_{fi} [analogous to Eqs. (37–39)] for the FSS detection/isolation

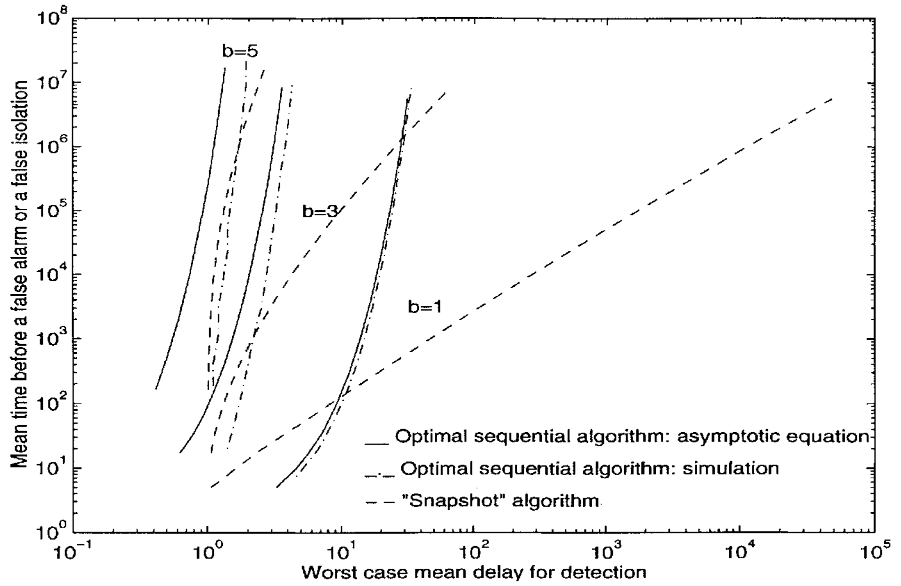

Fig. 4 Comparison between the optimal sequential fault detection/isolation algorithm and the snapshot algorithm: the mean time before a false alarm or a false isolation $\bar{T}$ of the optimal sequential algorithm, asymptotic formula, simulation, and the mean time before a false alarm or a false isolation $\bar{T}$ of the snapshot algorithm are plotted as functions of the worst-case mean detection/isolation delay $\bar{\tau}^*$.

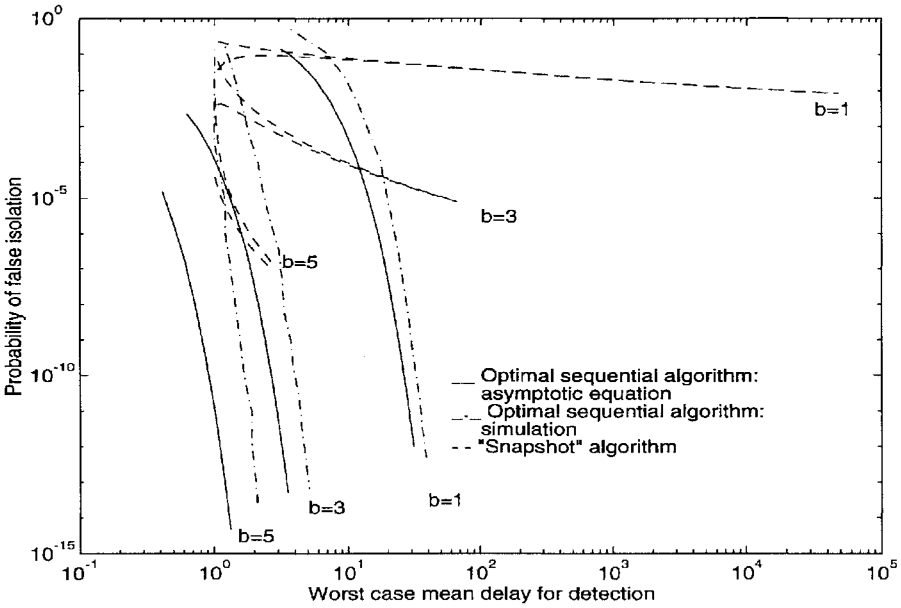

Fig. 5 Comparison between the optimal sequential and snapshot algorithms: the probability of false isolation $\beta_{\rm fi}$ of the optimal sequential algorithm, asymptotic formula, simulation, and lower and upper bounds for the probability of false isolation $\beta_{\rm fi}$ of the snapshot algorithm are plotted as functions of the worst-case mean detection/isolation delay $\bar{\tau}^*$.

algorithms in the case of receiver autonomous integrity monitoring. Hence, to estimate the performances of these algorithms it is necessary to use a numerical integration method.

Sequential Algorithm vs Snapshot: Criterion A

Let us consider the station-based integrity monitoring. We assume that $(\sigma_\psi/\sigma_\varphi) \gg 1$ and only positive jerks can arise in model (25). These simplifications are necessary to compute the performance indices of the snapshot algorithm. We fix a given value $\bar{T}_{\rm fai} = \min\{\bar{T}_{\rm fa}, \bar{T}_{\rm fi}\}$ and compute $\bar{\tau}^*(\bar{T}_{\rm fai})$ for the snapshot and sequential algorithms. For the sequential algorithm, we also use the Monre Carlo simulation to compute $\bar{\tau}^*$ for this value of $\bar{T}_{\rm fai}$. The results of this comparison are given in Fig. 4 when the SNR is $b = \upsilon/\sigma_\psi = 1, 3, 5$ and $n = 4$. This figure shows that the worst-case mean delay $\bar{\tau}^*$ of the optimal sequential algorithm is significantly lower than the $\bar{\tau}^*$ of the snapshot algorithm when the value of $\bar{T}_{\rm fai}$ is great, especially, for small values of the SNR b. We assume that the sampling period is equal to 0.6 s; if $b = 1$ and $\bar{T}_{\rm fai} = 10^7 \sim 0.2$ year, then the mean delay for detection of the optimal sequential algorithm is equal to $\simeq$19.2 s but the mean delay for detection of the snapshot algorithm is equal to $\simeq$12.2 h! Nevertheless, in the case of a large SNR ($b \geq 5$) or when the value of $\bar{T}$ is relatively small the optimal sequential and snapshot algorithms are equally efficient. The explanation of these results lies in the fact that the sequential algorithm (26), (27) is optimal in the sense of criterion A only asymptotically (i.e., when $\bar{T} \to \infty$).

Criterion B

Because the only tuning parameter of the snapshot algorithm is the threshold h it is impossible to choose between the arbitrary values of $\bar{T}_{\rm fa}$ and $\beta_{\rm fi}$. For this reason we compare the preceding algorithms in the following manner. For the snapshot algorithm we fix a mean time before a false alarm $\bar{T}_{\rm fa}$ and compute $\bar{\tau}^*$, lower and upper bounds for $\beta_{\rm fi}$ by using Eqs. (37–39). For the sequential algorithm, we use the asymptotic equation (36) to compute $\bar{\tau}^*$ for this value of $\bar{T}_{\rm fa}$. It follows from Eq. (36) the optimal choice of the thresholds is $h_d = \frac{1}{2}h_i$. The probabilities $\beta_{\rm fi}$ as functions of $\bar{\tau}^*$ for both the algorithms are shown in Fig. 5 when $b = 1, 3, 5$

and $n = 4$. Exactly as in the preceding case we estimate the worst-case mean detection delay $\bar{\tau}^*$ by using the simulation of the sequential algorithm. This figure show that exactly as in the preceding case the sequential algorithm has a significant advantage over the snapshot algorithm when the SNR is small or when β_{fi}^{-1} is large.

Sequential Algorithm vs FSS Competitor

We use criterion A for this comparison. In Sec. III it was shown that the sequential algorithm is asymptotically twice as good as the FSS competitor in the case of detection problem. It follows from Eqs. (35) and (37–39) that the sequential algorithm is also asymptotically twice as good as the FSS algorithm in the case of detection/isolation problem. (See proofs and details in Ref. 8.)

V. Simulation and Experiments with GPS Data

This section describes the results of statistical simulation and experiments with GPS data. Let us discuss the following GPS scenario. We assume a surface user with elevation mask angle of 7.5 deg. The user is located at Nantes, France (June 30, 1993, $\simeq$11 h 34 min universal time). Six satellites are visible (the satellites of block 1 are not considered). The sampling period is equal to 0.6 s. The satellite elevation angles $\mathcal{E}$ and azimuths $\mathcal{A}$ (in degrees) are

$$\begin{aligned} \mathcal{E} &= (9; 47; 45; 8; 54; 11) \\ \mathcal{A} &= (52; 147; 298; 246; 214; 305) \end{aligned} \tag{42}$$

Range errors [see Eq. (40)] are modeled as normally distributed, e.g., $\omega_t \sim \mathcal{N}(\boldsymbol{\theta}, \sigma^2 I_6)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_6)^T$, with $\sigma = 4$ m and θ_i ranging from -1 to 1 m with uniform probability. Note here that the preceding range error model is similar to the model described in Ref. 1. Channel degradations are modeled as additional biases of 25, 50, 75, and 100 m, respectively. The fourth column of Table 1 shows the radial error $\delta\boldsymbol{R} = \sqrt{(\delta x_u^2 + \delta y_u^2 + \delta z_u^2)}$ in user's fix, where δx_u, δy_u, and δz_u are the x, y, and z components of user's error resulting from the given biases. The first row of Table 1 shows the radial error without fault (only from θ).

The goal of this simulation is to compare the optimal sequential fault detection/isolation algorithm [see Eqs. (26), (27), and (41)] with the algorithm described in Ref. 1. The last row is a method of comparing the sum of squares of residual errors $SSE = \boldsymbol{e}_t^T \boldsymbol{e}_t$, where $\boldsymbol{e}_t = \Pi \boldsymbol{Y}_t$, using all satellites in view with a given threshold; if SSE is greater than this threshold, a fault has been detected. The isolation step consists in computing the sums of squares of residual errors, $SSE_1, SSE_2, \dots, SSE_n$, for n subsets of $n-1$ satellites. The sum $SSE_i = \boldsymbol{Y}_t^T(i)\Pi_i \boldsymbol{Y}_t(i)$, where $\Pi_i = I_{n-1} - H_i(H_i^T H_i)^{-1} H_i^T$, is computed under the assumption that the ith satellite is omitted [hence, the vector $\boldsymbol{Y}_t(i)$ and the matrix H_i are obtained from $\boldsymbol{Y}_t$, and H_u, respectively, by omitting the ith row]. The satellite channel, which is omitted from the subset, gives the smallest SSE_i and is identified as the failed one. The alarm time $\tilde{N}$ and the final decision $\tilde{\nu}$ of this snapshot algorithm are given by

$$\begin{aligned} \tilde{N} &= \inf\left\{ t \ge 1 : \sqrt{\frac{\boldsymbol{e}_t^T \boldsymbol{e}_t}{n-4}} \ge h\sigma \right\} \\ \tilde{\nu} &= \arg\min_{1 \le i \le n} \left\{ \boldsymbol{Y}_t^T(i)\Pi_i \boldsymbol{Y}_t(i) \mid t = \tilde{N} \right\} \end{aligned} \tag{43}$$

Table 1 Snapshot algorithm

Bias, m	Snapshot algorithm $\bar{\tau}$, s	Snapshot algorithm $\tilde{P}(\nu \ne l)$	δR before detection, m	δR after false alarm/isolation, m
0	—	—	0–3	0–6
25	206	0.33	10–33	10–76
50	4.2	0.2	21–63	22–147
75	0.63	0.1	33–93	36–218
100	0.6	6×10^{-2}	44–123	48–289

The SSE normalized by variance σ^2 has a noncentral χ^2 distribution with $n-4$ degrees of freedom and the noncentrality parameter $\lambda = (1/\sigma^2)\boldsymbol{\theta}^T \Pi \boldsymbol{\theta}$. For a given vector $\boldsymbol{\theta}$ the conditional mean time before a false alarm is defined as $\bar{T}(\boldsymbol{\theta}) = \{P[\chi^2_{n-4,\lambda} \ge h^2(n-4)]\}^{-1}$. The empirical mean time before a false alarm is defined as

$$\tilde{T} = \left(\frac{1}{\eta}\right) \sum_{p=1}^{\eta} \bar{T}(\boldsymbol{\theta}_p)$$

where $\boldsymbol{\theta}_p$ is a pseudorandom bias vector generated in the pth statistical experiment. For each value of h, there were $\eta = 10^4$ repetitions performed to estimate $\tilde{T}$. We assume that an acceptable level of false alarms is $\tilde{T} = 0.6 \cdot 10^6$ s $\simeq 167$ h. This value of $\tilde{T}$ is achieved when the threshold is $h = 3.75$. Table 1 presents the results of detection and isolation when the snapshot algorithm is implemented with $h = 3.75$ for biases of 25, 50, 75, and 100 m. The empirical estimates of the mean detection delay and the probability of false isolation are defined by

$$\bar{\tau} = \frac{1}{6}\sum_{i=1}^{6} \bar{\tau}_i, \qquad \tilde{P}(\nu \ne l) = \frac{1}{6}\sum_{i=1}^{6} \tilde{P}_i(\nu \ne i)$$

where $\bar{\tau}_i$ and $\tilde{P}_i(\nu \ne i)$ are empirical estimates of the mean detection delay and probability of false isolation when the ith channel is contaminated. We assume that all faults are equiprobable. For each type of fault, $\eta = 10^4$ repititions were performed. As mentioned in Ref. 1, "the primary objective of a large group of GPS users is to minimize the error in the overall navigation solution." Let us analyze Table 1 from this angle. First of all, it is obvious that the consequence of a false isolation is much more dangerous than the consequence of a false alarm. (See columns 4 and 5 of Table 1; the results of false alarms are presented in the first row.) The false isolations will seriously degrade the accuracy of the navigation solution. From this point of view the isolation performances of the snapshot algortihm are insufficient. For example, if the bias is 50 m, the most dangerous situations are $i = 3$, $\nu = 4$ (e.g., satellite channel 3 is contaminated, but channel 4 is mistakenly removed from the navigation solution because of a false isolation) and $i = 4$, $\nu = 3$. These false isolations lead to the additional radial errors $\delta R \simeq 136$–147 m and $\delta R \simeq 89$–100 m, respectively. The empirical probabilities of theses events are $\tilde{P}_3(\nu = 4) = 0.1$ and $\tilde{P}_4(\nu = 3) = 0.18$, respectively. Increasing the threshold h does not improve the situation. Hence, to improve the isolation properties of detection/isolation algorithm (43), it is necessary to use several sets of GPS signals instead of a single set. Let us define the following FSS algorithm:

$$\tilde{N} = \inf_{j \ge 1} \left\{ jm : \sqrt{\frac{\left(\sum_{t=(j-1)m+1}^{jm} \boldsymbol{e}_t\right)^T \left(\sum_{t=(j-1)m+1}^{jm} \boldsymbol{e}_t\right)}{m(n-4)}} \ge h\sigma \right\} \tag{44}$$

$$\begin{aligned} \tilde{\nu} = \arg\min_{1 \le i \le n} \Bigg\{ & \left(\sum_{t=(j-1)m+1}^{jm} \boldsymbol{Y}_t(i) \right)^T \\ & \times \Pi(i) \left(\sum_{t=(j-1)m+1}^{jm} \boldsymbol{Y}_t(i) \right) \Bigg| j = \frac{\tilde{N}}{m} \Bigg\} \end{aligned} \tag{45}$$

The additional bias of 75 m was chosen as an assumed fault magnitude. We require that the probability of false isolation should be $\tilde{P}(\nu \ne l) \le 10^{-4}$ (under the assumption that $\upsilon = 75$ m) and the level of false alarms should be $\tilde{T} \ge 0.6 \cdot 10^6$ s. The mean detection delay $\bar{\tau}$ of algorithm (44) and (45) is a function of the sample size m and the threshold h (tuning parameters of the FSS algorithm). Hence, the optimal choice of the parameters m and h is reduced to minimizing the mean detection delay $\bar{\tau}(m, h)$ under the constraints $\tilde{P}(\nu \ne l) \le 10^{-4}$, $\tilde{T} \ge 0.6 \cdot 10^6$ s, and $\upsilon = 75$ m. The optimal values of these parameters are $m = 40$ and $h = 11.5$. The following

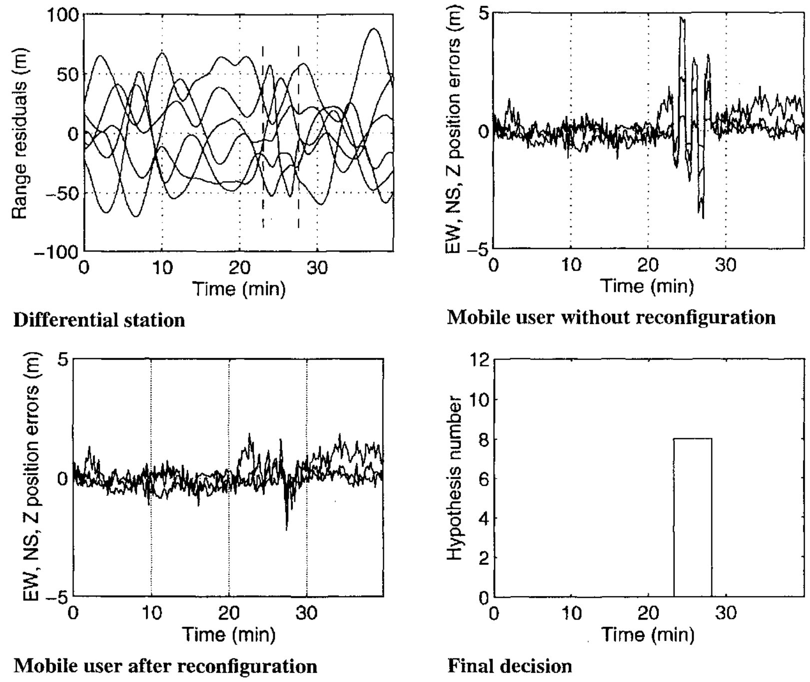

Fig. 6 Simulation of the detection/isolation at a differential station and the reconfiguration at a user's receiver. The start (time = 23 min) and the end (time = 27 min 36 s) of the global degradation of satellite 4 in range residuals are indicated by the vertical dotted lines (upper left). (Courtesy of the SERCEL-FRANCE, Nantes, France.)

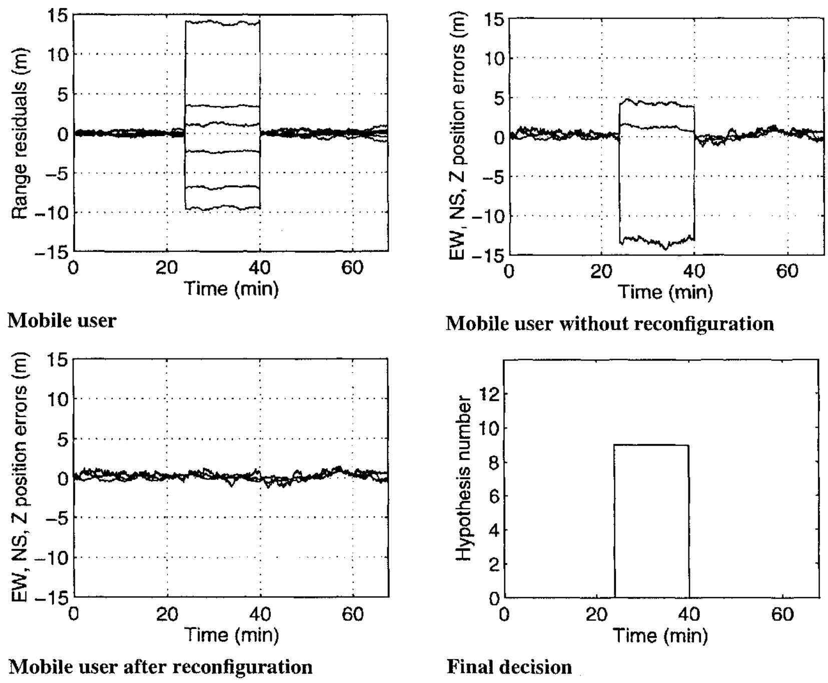

Fig. 7 Simulation of the detection, isolation, and reconfiguration at a user's receiver. The local degradation of satellite 5 in range residuals is started at 24 min and is finished at 40 min (upper left). (Courtesy of the SERCEL-FRANCE, Nantes, France.)

simplified recursive version of the sequential algorithm (26), (27), and (41) is used for the comparison:

$$N = \min\{N^1, \dots, N^{2n}\}$$

$$N^l = \inf\left\{t \geq 1 : \min_{0 \leq j \neq l \leq 2n} [g_t(l, j) - h_{lj}] \geq 0\right\}$$

$$g_t(l, j) = g_t(l, 0) - g_t(j, 0)$$

$$g_t(l, 0) = \left(g_{t-1}(l, 0) + \frac{(-1)^l \tilde{\upsilon} e_{t,i}}{\sigma^2 \sqrt{p_{ii}}} - \frac{\tilde{\upsilon}^2}{2\sigma^2}\right)^+$$

where $i = [(l+1)/2]$, $g_0(1, 0) = \cdots = g_0(2n, 0) = 0$, and h_{lj} are given by Eq. (28). To reach the required levels of false alarms and false isolations, the following tuning parameters of the preceding sequential algorithm have been chosen: $\tilde{\upsilon} = 25$, $h_d = 5.5$, and $h_i = 7.5$. Table 2 presents the results of the comparison between the FSS algorithm and the sequential algorithm, for biases of 25, 50, 75, and 100 m. It is easy to see that the proposed sequential algorithm is much more efficient than the FSS algorithm for all fault magnitudes.

The results of two experiments with GPS data are presented in Figs. 6 and 7. The first experiment is the detection/isolation of global degradation at a differential station. We assume that this differential station broadcasts the pseudorange corrections (2) with the update rate of 12 s. The pseudorange errors at the user's position are modeled artificially by using the following autoregressive equation: $\gamma_t = 0.99 \cdot \gamma_{t-1} + \varsigma_t$, where $\varsigma_t \sim \mathcal{N}(0, \sigma_\varsigma^2)$ and $\sigma_\gamma = 2$ m. The

Table 2 FSS and sequential algorithms

Bias, m	FSS algorithm $\bar{\tau}$, s	FSS algorithm $\tilde{P}(\nu \neq l)$	Sequential algorithm $\bar{\tau}$, s	Sequential algorithm $\tilde{P}(\nu \neq l)$	δR before detection, m	δR after false isolation, m
25	$\simeq 0.5 \times 10^6$	3.4×10^{-2}	100	4.7×10^{-2}	10–33	10–76
50	33.4	1.7×10^{-3}	4.4	1.5×10^{-3}	21–63	22–147
75	30.2	$\leq 10^{-4}$	2.8	$\leq 10^{-4}$	33–93	36–218
100	28.6	$\leq 10^{-4}$	1.8	$\leq 10^{-4}$	44–123	48–289

global degradation of satellite 4 ($\mathcal{E} = 8$ deg, $\mathcal{A} = 246$ deg) [see Eq. (42)] is simulated as a series of alternating jerks. The results of this degradation, after a triple integration, are presented in Fig. 6; see range residuals at the differential station (upper left) EW, NS, and Z position errors (from 1 to 5 m) of a mobile user (upper right). A new feature of this experiment is the fault detection/isolation algorithm connected with the χ^2-CUSUM algorithm to detect the instant of a fault's disappearance. The final decision ν of this couple is presented in Fig. 6 (lower right) as a function of time. The proposed algorithm is able to detect/isolate and properly eliminate the impact of this fault. [See, Fig. 6 (lower left), EW, NS, and Z position errors of a mobile user after reconfiguration.] The second experiment is the detection/isolation of local degradation by using the receiver autonomous integrity monitoring. We assume the same GPS scenario. The local fault of satellite 5 ($\mathcal{E} = 54$ deg, $\mathcal{A} = 214$ deg) is modeled as a bias of 25 m. The proposed fault detection/isolation algorithm permits the user to eliminate the impact of this fault (see Fig. 7).

VI. Conclusion

A new optimal sequential approach to the GPS/DGPS integrity monitoring (fault detection and isolation) was proposed. Simple models of GPS/DGPS channels were suggested and two particular schemes of integrity monitoring, autonomous receiver and differential station based, were discussed. The proposed sequential approach was compared with the fixed size sample approach (particularly, snapshot) by using asymptotic equations and Monte–Carlo simulations. The essential advantage of the proposed sequential algorithm over the snapshot algorithm was shown.

Acknowledgments

The author is grateful to J. P. Barboux, G. Bonin, and D. A. Charrier [Société d'Études, Recherches et Constructions Électroniques (SERCEL), France] for very fruitful discussions and kind help. He is also very grateful to W. M. Lear (Charles Stark Draper Laboratory, NASA Johnson Space Center) for his help in the problem of the GPS range bias models.

References

[1]Parkinson, B. W., and Axelrad, P., "Autonomous GPS Integrity Monitoring Using the Pseudorange Residual," *Navigation*, Vol. 35, No. 2, 1988, pp. 255–274.

[2]Sturza, M. A., "Navigation System Integrity Monitoring Using Redundant Measurements," *Navigation*, Vol. 35, No. 4, 1988–89, pp. 483–501.

[3]Lee, Y. C., "Receiver Autonomous Integrity Monitoring (RAIM) Capability for Solemeans GPS Navigation in the Oceanic Phase of Flight," *IEEE AES Magazine*, May 1992, pp. 29–36.

[4]Misra, P. N., "Integrated Use of GPS and GLONASS in Civil Aviation," *Lincoln Laboratory Journal*, Vol. 6, No. 2, 1993, pp. 231–247.

[5]Lear, W. M., Montez, M. N., Lonnie, M. R., and Zyla, L. V., "The Effect of Selective Availability on Orbit Space Vehicles Equipped with SPS GPS Receivers," *Proceedings of the Institute of Navigation*, 1992, pp. 825–840.

[6]Lorden, G., "Procedures for Reacting to a Change in Distribution," *Annals of Mathematical Statistics*, Vol. 42, No. 6, 1971, pp. 1897–1908.

[7]Page, E. S., "Continuous Inspection Schemes," *Biometrika*, Vol. 41, No. 1, 1954, pp. 100–115.

[8]Nikiforov, I. V., "Two Strategies in the Problem of Change Detection and Isolation," *IEEE Transactions on Information Theory* (to be published).

[9]Nikiforov, I. V., "On the First-Order Optimality of an Algorithm for Detection of a Fault in the Vector Case," *Automation and Remote Control*, Vol. 55, No. 1, 1994, pp. 68–82.

[10]Basseville, M., and Nikiforov, I. V., *Detection of Abrupt Changes. Theory and Applications*, Information and System Sciences Series, Prentice–Hall, Englewood Cliffs, NJ, 1993.

[11]Nikiforov, I. V., "A Generalized Change Detection Problem," *IEEE Transactions on Information Theory*, Vol. IT-41, No. 1, 1995, pp. 171–187.

[12]Nikiforov, I. V., "On Two New Criteria of Optimality for the Problem of Sequential Change Diagnosis," *Proceedings of the American Control Conference* (Seattle, WA), 1995, pp. 97–101.